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Matricial Norms Over Cones

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ABSTRACT

The notion of a vector-valued norm is extended to norms taking values in a partially ordered finite dimensional space. Sufficient conditions are given for the existence of the dual norm and the matrix norm subordinate to a vector norm.

In a number of recent papers the notion of a norm has been extended to a mapping which satisfies the usual definition of a norm except that it takes its values in the nonnegative orthant (cf. [2], [4], [5], [7], [8], [9]).

Let V be a Kantorovich space (cf. [6]) of dimension k. That is, V is a real vector space of dimension k which is partially ordered by a closed convex full pointed cone K (cf. [1], [10], [12]). The dual space V^* is also a Kantorovich space under the partial order induced by the *dual cone*

$$K^* = \{f \in V^* | fx \geqslant 0, x \in K\}.$$

We shall use the following notation relative to the partial order in K (or K^*):

$$x \geqslant y \leftrightarrow x - y \in K,$$

$$x > y \leftrightarrow x \geqslant y \quad \text{and} \quad x \neq y,$$

$$x \gg y \leftrightarrow x - y \in \operatorname{Int} K.$$

DEFINITION 1. *A* vectorial norm *is a function* $p : \mathbf{C}^n \to K$ *which for* $x, y \in \mathbf{C}^n$, $\alpha \in \mathbf{C}$ *satisfies*

$$p(x) \geqslant 0, \qquad p(x) = 0 \quad \textit{iff} \quad x = 0, \tag{1}$$

$$p(\alpha x) = |\alpha| p(x), \tag{2}$$

$$p(x + y) \leqslant p(x) + p(y). \tag{3}$$

Let p be a vectorial norm and let $f \in K^*$. Define the function φ_f by $\varphi_f(x) = fp(x)$.

φ_f is a seminorm on $\mathbf{C}^n$. If ψ is any seminorm on $\mathbf{C}^n$ let

$$W = \{f \in K^* | \psi(x) \leqslant \varphi_f(x) = fp(x)\}.$$

LEMMA 1. *For all $f \gg 0$, φ_f is a norm.*

Proof. $\varphi_f(x) = 0 \leftrightarrow fp(x) = 0 \leftrightarrow p(x) = 0 \leftrightarrow x = 0$. ∎

Since K^* is full we have

COROLLARY. *Any vectorial norm is continuous.*

LEMMA 2.

$$\forall f \gg 0, \quad \exists \lambda > 0 \; \forall x \in \mathbf{C}^n$$

$$\psi(x) \leqslant \lambda[fp(x)] = \lambda \varphi_f = \varphi_{\lambda f}(x).$$

Proof. It is enough to establish the inequality for x in

$$B = \{y \in \mathbf{C}^n | y^* y = 1\}.$$

B is compact so $\psi(B)$ has an upper bound λ_0. Since φ_f is a norm, $0 \notin \varphi_f(B)$ and so $\varphi_f(B)$ has a lower bound $\lambda_1 > 0$. Choose $\lambda \geqslant \lambda_0/\lambda_1$. The last equality is clear.

COROLLARY. $W \neq \emptyset$. *In fact,* $\forall f \gg 0$, $\exists \lambda > 0$, $\lambda f \in W$.

It is also easily seen that W is convex and that $f \in W$, $g \geqslant f$ imply $g \in W$. Thus $W \supset K^* + f$ for any $f \in W$. ∎

DEFINITION 2. *Let p be a vectorial norm. If $x \in \mathbf{C}^n$, let*

$$|x| = \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix}.$$

(*a*) *p is monotone iff* $|x| \leqslant |y| \Rightarrow p(x) \leqslant p(y)$.
(*b*) *p is absolute iff* $p(x) = p(|x|)$.

THEOREM 1. *A vectorial norm p is monotone iff it is absolute.*

Proof. $p(x) \leqslant p(y)$ iff $\forall f \in K^*$, $\varphi_f(x) \leqslant \varphi_f(y)$, that is iff φ_f is a monotone norm for all $f \gg 0$. However, p is absolute iff $\forall f \gg 0$, φ_f is an absolute norm. Now we appeal to Theorem 2 of [3] to see that φ_f is monotone iff it is absolute. The proposition is established. ■

DEFINITION 3. *Let*

$$W(y) = \{f \in K^* \mid |y^*x| \leqslant \varphi_f(x) = fp(x)\}.$$

The vectorial norm p is regular *if for each $y \in \mathbf{C}^n$ there is an $f_0 \in K^*$ such that $W(y) = f_0 + K^*$. If p is regular define the* dual *to p by*

$$p^D(y^*) = f_0.$$

If K is the nonnegative orthant, then by the results of [9] we know that this use of regular corresponds to that of Robert and Deutsch.

LEMMA 3. *If x_0 is taken so that*

$$\max\{|y^*| \mid x \in B\} = |y^*x_0|$$

(for a fixed $y \in \mathbf{C}^n$) and if p is regular, then

$$|y^*x_0| = p^D(y^*)p(x_0).$$

Proof. The existence of such an x_0 is common knowledge. If $p^D(y^*)p(x_0) > |y^*x_0|$, then for $\varepsilon > 0$ suitably small we have

$$(1-\varepsilon)p^D(y^*)p(x) = p^D(y^*)p(x) - \varepsilon p^D(y^*)p(x) > |y^*x_0| \geqslant |y^*x|$$

for all $x \in B$. So $(1-\varepsilon)p^D(y^*) \in W(y)$ which contradicts the minimality of $p^D(y^*)$ in $W(y)$. ■

THEOREM 2. *If p is a regular vectorial norm, then p^D is a norm.*

Proof. As in Lemma 3 we shall use the fact that $f_0 = p^D(y^*)$ is the unique minimal element of $W(y)$. Condition (1) of Definition 1 is immediate. Let $f_1 = p^D(\bar{\alpha}y^*)$, for $\alpha \neq 0$. Then for any x we have

$$|y^*x| \leqslant |\alpha|^{-1} f_1 p(x), \qquad |\bar{\alpha} y^* x| \leqslant |\alpha| f_0 p(x).$$

So $|\alpha|^{-1} f_1 \in W(y)$ and $|\alpha| f_0 \in W(\alpha y)$. Thus $f_1 = |\alpha| f_0$. Finally let $f_1 = p^D(y_1^*)$, $f_2 = p^D(y_2^*)$, $f_3 = p^D(y_1^* + y_2^*)$. Then for all x

$$f_1 p(x) + f_2 p(x) \geqslant |y_1^* x| + |y_2^* x| \geqslant |(y_1 + y_2)^* x|.$$

Thus $f_1 + f_2 \in W(y_1^* + y_2^*)$ and by the minimality of f_3

$$p^D(y_1^*) + p^D(y_2^*) = f_1 + f_2 \geqslant f_3 = p^D(y_1^* + y_2^*). \quad \blacksquare$$

DEFINITION 4. *Let p be regular. p is* positively oriented *iff for any $x \neq 0$ there is a y such that $|y^*x|$ is maximal and $p^D(y^*) \gg 0$.*

THEOREM 3. *If p is regular and p^D is both regular and positively oriented, then $p^{DD} = p$.*

Proof. We let

$$W(x) = \{a \in K \mid |y^*x| \leqslant p^D(y^*)a\}.$$

Observe first that $p(x) \in W(x)$ since for all y, and all $f \in W(y)$, $|y^*x| \leqslant f p(x)$. Thus for all y we have

$$|y^*x| \leqslant p^D(y) p(x).$$

Since p^D is regular there is an $a_0 \in K$ such that $W(x) = a_0 + K$. Choose y so that $|y^*x|$ is maximal and $p^D(y^*) \gg 0$. Thus for any $a \in W(x)$ by Lemma 3

$$|y^*x| = p^D(y^*) p(x) \leqslant p^D(y^*) a.$$

In particular for a_0 we have

$$p^D(y^*)(p(x) - a_0) \leqslant 0.$$

But $p^D(y) \gg 0$ and $p(x) - a_0 \geqslant 0$, so $p(x) = a_0$. Hence $p^{DD} = p$. $\blacksquare$

The cone K induces a partial order in $\mathrm{End}(V)$, the space of endomorphisms of V, by

$$M \geqslant 0 \Leftrightarrow MK \subseteq K.$$

The nonnegative cone of $\mathrm{End}(V)$ is denoted by Π.

DEFINITION 5. *Let* $\mu : \mathbf{C}^{n,n} \to \mathrm{End}(V)$ *be a vectorial norm on the set of* $n \times n$ *matrices. It is a* matricial norm *iff*

$$\mu(AB) \leqslant \mu(A)\mu(B). \tag{4}$$

In looking for a matricial norm which deserves to be considered the bound norm subordinate to a given regular vectorial norm there are three sets for each $A \in \mathbf{C}^{n,n}$ which arise rather naturally. These are

$$W_0(A) = \{M \in \Pi \,|\, |y^*Ax| \leqslant p^D(y^*)Mp(x)\},$$

$$W_1(A) = \{M \in \Pi | p(Ax) \leqslant Mp(x)\},$$

$$W_2(A) = \{M \in \Pi | p^D(y^*A) \leqslant p^D(y^*)M\}.$$

THEOREM 4. *Let* p *be a regular vectorial norm, and let* p^D *be regular and positively oriented. If for each* $A \in \mathbf{C}^{n,n}$ *there is an* $M_0 \in \Pi$ *such that* $W_0(A) = M_0 + \Pi$, *then*

(1) $W_0 = W_1 = W_2$,

(2) $\mathrm{lub}_p(A) = M_0$ *defines a matricial norm for which* $\mathrm{lub}_p(I) \leqslant I$, *and* $\mathrm{lub}_p(I) = I$ *if there is an* x *for which* $p(x) \gg 0$.

As a consequence of statement (1) lub_p *is compatible with* p *and* p^D; *that is,* $p(Ax) \leqslant \mathrm{lub}_p(A)p(x)$ *and* $p^D(y^*A) \leqslant p(y^*)\,\mathrm{lub}_p(A)$.

Proof. Ad(1). Clearly $W_0 \supset W_1$, $W_0 \supset W_2$. Let $M \in W_0$. Then

$$|y^*Ax| \leqslant p^D(y^*)Mp(x), \qquad |y^*(Ax)| \leqslant p^D(y^*)p(Ax). \tag{$*$}$$

But $p^{DD} = p$, so from ($*$) we have $M_0p(x) \in W(Ax)$. Thus $p(Ax) \leqslant M_0p(x) \leqslant Mp(x)$, since $M_0 \leqslant M$. Hence $M \in W_1$ and $W_0 = W_1$. The proof that $W_0 = W_2$ is similar.

Ad(2). lub_p is clearly a vectorial norm. Let $A, B \in \mathbf{C}^{n,n}$, and let $M_1 = \mathrm{lub}_p(B)$.

$$|y^*ABx| \leqslant p^D(y^*)M_0p(Bx) \leqslant p^D(y^*)M_0M_1p(x) \quad \text{by} \quad (1).$$

So lub_p is a matricial norm. Suppose $\mathrm{lub}_p(I) = N$. $|y^*x| = |y^*Ix| \leqslant p^D(y^*)p(x)$. So $\mathrm{lub}_p(I) = N \leqslant I$. For suitable y and x, $p^D(y^*) \gg 0$ and

$$p^D(y^*)p(x) = |y^*x| = |y^*Ix| = p^D(y^*)Np(x).$$

Since $p^D(y^*) \gg 0$ we have

$$p^D(y^*)(N - I)p(x) = 0 \Rightarrow (N - I)p(x) = 0.$$

But if $p(x) \gg 0$, then $N - I = 0$.

The final statement is a rephrasing of the equality $W_0 = W_1 = W_2$. ■

This result says in (cf. [8]) that lub_p is a majorant for p. Both Robert and Stoer [11] employ the class of B-matrices in studying minorants. In this direction we have the machinery to obtain a couple of their lemmas. First recall a definition from [1].

Definition 6. *A matrix N admits a* completely regular splitting *iff $N = B - C$ where $B, B^{-1} > 0$ and $C \geqslant 0$. If N admits a completely regular splitting, then N is an M-*matrix *iff $N^{-1} > 0$.*

Definition 7. *A matrix N is called a B-*matrix *iff the set*

$$\mathscr{D}(N, \omega) = \{u \geqslant 0 | Nu \leqslant \omega\}$$

is bounded for all $\omega \geqslant 0$.

Theorem 5. *Let N admit a completely regular splitting. Then N is an M-matrix iff it is a B-matrix.*

Proof. If N is an M-matrix, then $N^{-1} > 0$. So

$$Nu \leqslant \omega \Rightarrow u \leqslant N^{-1}\omega,$$

and thus $\mathscr{D}(N, \omega)$ is bounded.

Conversely, suppose N is a B-matrix. Let $N = B - C$ be a completely regular splitting. B^{-1} is a one to one map which takes the boundary of K onto the boundary of K and the interior of K onto the interior of K. Thus

$$\{u \geqslant 0 | Nu \leqslant \omega\}$$

is bounded iff

$$\mathscr{S} = \{u \geqslant 0 | (I - B^{-1}C)u \leqslant B^{-1}\omega\}$$

is bounded. Further N is a B-matrix iff $\mathscr{S}$ is bounded for all $\omega \geqslant 0$. Let ρ be the Perron root of $B^{-1}C$ with $v \geqslant 0$ a corresponding eigenvector, and let ρ_1 be the Perron root of B^{-1} with $\omega \geqslant 0$ a corresponding eigenvector. If $\rho \geqslant 1$, then

$$v - B^{-1}Cv = (1 - \rho)v \leqslant 0 < \rho_1 \omega = B^{-1}\omega.$$

Thus $v \in \mathscr{S}$, and for any $\alpha > 0$, $\alpha v \in \mathscr{S}$. Thus $\rho < 1$ and by Lemma 1 of [10] N is an M-matrix. ∎

THEOREM 6. *The following are equivalent.*

(1) N *is a* B-*matrix.*
(2) $Nu \leqslant 0, \quad u \geqslant 0 \Rightarrow u = 0.$
(3) $\exists f \geqslant 0, fN \gg 0.$

Proof. Statement (1) $\Rightarrow$ Statement (2) is trivial since (2) is the statement that $\mathscr{D}(N, 0) = \{0\}$. Statement (2) $\Rightarrow$ Statement (3) is an immediate consequence of problem 34 of chapter 8 of [6]. To prove that statement (3) $\Rightarrow$ statement (1) suppose $fN \gg 0$. Then we have

$$\mathscr{D}(N, \omega) \subseteq \{u \geqslant 0 | 0 \leqslant fNu \leqslant f\omega\},$$

and the latter set is bounded because $fN \gg 0$; i.e., $u \geqslant 0$ and $fNu = 0 \Rightarrow u = 0$. ∎

In [5] Deutsch obtained several results which compared the spectral radius $r(A)$ of $A \in \mathbf{C}^{n,n}$ with $r[\mu(A)]$ where μ is any matricial norm. In particular his Propositions 4, 6, 7, and 9 carry over to matricial norms in the present sense after we prove the analog of his Proposition 5.

THEOREM 7. *If* μ *is a matricial norm, then* μ *is a continuous function. Further if* $\mu(A_n) \to 0$, *then* $A_n \to 0$, *where convergence is in the unique topologies which make* $\mathbf{C}^{n,n}$ *and* End(V) *topological vector spaces.*

Proof. The first remark follows from the corollary to Lemma 1. If $\mu(A_n) \to 0$, then for any $f \in K^*$, $x \in K$ we have $f\mu(A_n)x \to 0$. However, if $f \gg 0$, $x \gg 0$ it is readily verified that the function $A_0 \to f\mu(A_0)x$ is a (scalar valued) norm, whence $\mu(A_n) \to 0$ implies $A_n \to 0$. ∎

Let $\mathscr{M}$ denote the set of all matricial norms from $\mathbf{C}^{n,n}$ to Π. If ν is any scalar valued multiplicative norm and $f \gg 0$, $x \gg 0$, then $xf \in \Pi$ and the function μ defined by

$$\mu(A) = \nu(A)xf$$

is a matricial norm. Now for any given A and for any $\varepsilon > 0$ we can find a norm ν depending on A and ε such that $\nu(A) \leqslant r(A) + \varepsilon$. But then

$$r[\mu(A)] = \nu(A) \leqslant r(A) + \varepsilon.$$

We have proved

THEOREM 9. *Let $A \in \mathbf{C}^{n,n}$. Then*

$$r(A) = \inf\{r(\mu(A))|\mu \in \mathscr{M}\}.$$

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